# Permutation and Combination

## Basics

### Definitions
- **Permutation** - ORDERED arrangement of objects
- **Combination** - UNORDERED selections of objects

### Permutation and Combination with DISTINCT objects

$n$ = the number of all objects

$r$ = the number of objects take out from $n$ objects to do arrangement and selection

<table>
<thead>
<tr>
<th>Type</th>
<th>Repetition Allowed?</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>r-Permutation</td>
<td>NO</td>
<td>$P(n, r) = \frac{n!}{(n-r)!}$</td>
</tr>
<tr>
<td>r-Permutation</td>
<td>YES</td>
<td>$n^r$</td>
</tr>
<tr>
<td>r-Combination</td>
<td>NO</td>
<td>$C(n, r) = \frac{n!}{r!(n-r)!}$</td>
</tr>
<tr>
<td>r-Combination</td>
<td>YES</td>
<td>$C(n + r - 1, r) = \frac{(n + r - 1)!}{r!(n - 1)!}$</td>
</tr>
</tbody>
</table>

### Permutations with INDISTINGUISHABLE objects

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

Different permutations of $n$ objects, where there are $n_1$ INDISTINGUISHABLE of objects of type 1, $n_2$ INDISTINGUISHABLE of objects of type 2, …, $n_k$ INDISTINGUISHABLE of objects of type $k$
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Example Questions

1. r-Permutation of n DISTINCT objects with NO REPETITION

If a repeated letter is not allowed in the string, how many 4-letter strings can be formed from the 26 uppercase English alphabets?

**Solution:** If any letter of a string changes, it makes a new string, so order matters. In an ORDERED counting, we use PERMUTATION. We are arranging 4 letter out of 26 alphabets, so \( n=26, r=4 \). Because NO REPETITION is allowed, if a letter is used in one slot of the string, it can’t be used for the other slots. The choices of each letter within the 4-letter string is \( 26 \cdot 25 \cdot 24 \cdot 23 \). The number of 4-letter strings of uppercase English alphabets will be

\[
P(26, 4) = \frac{26!}{(26-4)!} = \frac{26!}{22!}
\]

2. r-Permutation of n DISTINCT objects with REPETITION

How many 4-letter strings can be formed from the uppercase English alphabet?

**Solution:** If any letter of a string changes, it makes a new string, so order matters. In ORDERED counting, we use PERMUTATION. We are arranging 4 letter out of 26 alphabets, so \( n=26, r=4 \). Because REPETITION is allowed, each letter can be any of the 26 alphabets. The choices of each letter within the 4-letter string is \( 26 \cdot 26 \cdot 26 \cdot 26 \). The number of 4-letter strings of uppercase English alphabets is \( 26^4 \)

3. Permutation with INDISTINGUISHABLE objects

How many different arrangements are there of the letters in the word "SUCCESS"?

**Solution:** If any letter of a string changes, it makes a new string, so order matters. In an ORDERED counting, we use PERMUTATION. We are RE-ARRANGING 7 letters within word "SUCCESS", so \( n=r=7 \). In word "SUCCESS", "S_3UC_1C_2ES_1S_2" and "S_2UC_2C_1ES_3S_1" are the same word. The two C’s and three S’s are INDISTINGUISHABLE, so we can’t use the methods with distinct objects. There are 2 INDISTINGUISHABLE C’s and 3 INDISTINGUISHABLE S’s, so \( n_1=2 \) and \( n_2=3 \). The number of different arrangement is

\[
\frac{n!}{n_1!n_2!} = \frac{7!}{2!3!}
\]
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4. r-Combination of n DISTINCT objects WITHOUT repetition

There is random picking of 5 numbers from 1 to 10, and each number can only be picked once. How many combinations can be formed?

Solution: For picking numbers, order doesn’t matter. In an UNORDERED counting, we use COMBINATION. We are selecting 5 numbers out of 10 numbers, so \( n=10, r=5 \). Because NO REPETITION is allowed, if a number is already picked, it can't be picked again. The number of 5-number combinations picking from 1 to 10 will be \( C(10,5) = \frac{10!}{(10-5)!} = \frac{10!}{5!} \)

5. r-Combination of n DISTINCT objects WITH repetition

Randomly picking 5 numbers from 1 to 10, and each number can be picked repeatedly. How many combinations can be formed?

Solution: For picking numbers, order doesn’t matter. In an UNORDERED counting, we use COMBINATION. We are selecting 5 numbers out of 10 numbers, so \( n=10, r=5 \). Because REPETITION is allowed, even though a number is already picked, it can be picked again. The number of combinations will be \( C(10,5) = \frac{10!}{(10-5)!} = \frac{10!}{5!} \)

Steps to solve the problems

When you see a problem, please ask yourself the following questions:

1. Does ORDER matter in counting?
2. What is the n and r?
3. Any INDISTINGUISHABLE objects?
4. Repetition allowed?

References: The following works were referred to during the creation of this handout: Discrete Mathematics and Its Applications. 7th ed. Kenneth H. Rosen.