Optimization Problems

Overview: The methods used for finding extreme values for functions have practical applications in many areas of life. For example, a traveler wanting to minimize transportation time. In solving such practical problems, one has to convert the word problem into a mathematical optimization problem and set up a function to be maximized or minimized.

Steps in solving optimization problems:

1. Understand the problem: Read the problem carefully to find out what the problem is asking. Then, underline the important pieces of information in the problem.

2. Draw a diagram (optional): It is always helpful to sum up the entire problem in a simple diagram so to prevent reading the problem repeatedly.

3. Introduce notion and express it in terms of variables: Assign a symbol to the quantity that has to be maximized or minimized. (Let us call it Z for now). Assign variable names to the unknowns and express Z as a function of those variables.

4. Try to express notion in terms of one variable: If Z has been expressed as a function of more than one variable (step 4), use the information in the problem to eliminate all but one of the variables and use that to express Z.

5. Differentiate function and equate it to zero to obtain critical points: Differentiate Z with respect to the variable you choose and equate it to zero to obtain the values of that variable (critical points).

6. Test critical points for max/min using the second derivative: To test whether the critical points are a max or min (concave down or up respectively), we take the second derivative of Z and plug in the critical points obtained in step 6 to see whether we get a positive value (minimum) or a negative value (maximum).

7. Use the required critical point to find the optimal answer: Once we know what critical points we are using, we plug that in Z to obtain the answer to our problem.

Now let us use these steps on a few examples!
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Problem 1: The sum of two positive numbers is 16. What is the smallest possible value of the sum of their squares?

SOLUTION:

STEP 1: We have,

a) Sum of two positive numbers is 16 (info)
b) Smallest possible value for sum of their squares (objective)

STEP 2: will be ignored since diagram is not possible for this problem.

STEP 3: Let us assume the numbers are \( x \) and \( y \) and the sum of their squares is \( S \).

We are given: \( x + y = 16 \) x > 0 and \( y > 0 \) (since they are positive)

We need to minimize: \( S = x^2 + y^2 \)

STEP 4: Since \( S \) is expressed in terms of both \( x \) and \( y \), we need to try to express \( S \) with only one variable. From the information given, we have \( x + y = 16 \), or \( y = 16 - x \).  

Putting this value of \( y \) in \( S \) we get, 

\[ S(x) = x^2 + (16-x)^2. \]

When we simplify this, \( S(x) = x^2 + x^2 - 24x + 256. \) Combining like terms we get, 

\[ S(x) = 2x^2 - 32x + 256 \] which we need to minimize.

STEP 5: Calculating the first derivative of \( S(x) \) we get, 

\[ S'(x) = 4x - 32 \]

To find the critical points, we need to equate \( S'(x) \) to zero.

\[ => 4x - 32 = 0 \] or \( 4x = 32 \) or \( x = 8 \).  

---------- (2)

STEP 6: To test whether this value gives us a minimum, we take out the second derivative: \( S''(x) = 4 \), which is positive. Thus, we know that we have minimized \( S \).

STEP 7: But, we still haven't completed the problem. We still need to find the other number, i.e. \( y \).

From (1), we have \( y = 16 - x \).

After putting in the \( x = 8 \) we found from (2) and solving for \( y \),

we get \( y = 8 \).

Therefore, the two positive numbers, which add up to 16 and, with the smallest possible value for the sum of their squares, are 8 and 8.
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Problem 2: A rectangular storage container with an open top needs to have a volume of 10 m$^3$. The length of its base is twice the width. Material for the base costs $10 per m$^2$. Material for the sides costs $6 per m$^2. Find the cost of the material for the cheapest container.

SOLUTION:

STEP 1: We have,
   a) Rectangular box with open top (info)
   b) Volume of the box is 10 m$^3$ (info)
   c) Length of the base is twice its width (info)
   d) Material for the base costs $10 / m^2$ and material for the sides costs $6 / m^2$ (info)
   e) The cost of the material for the cheapest container (objective)

STEP 2: 

STEP 3: Let us assume the height of the box is $h$ and the width of the box is $x$. Then from the information given, we obtain the length of the box as $2x$. Let the cost of the box be $C$.

Cost = $10(area of base) + $6(area of 2 long sides) + $6(area of 2 short sides)

which gives,

Cost = $10(2x^2) + (2 \times 6 \times x \times h) + (2 \times 6 \times 2x \times h)$

= $20x^2 + 12xh + 24xh$

= $20x^2 + 36xh$

STEP 4: Since $C$ is expressed in terms of both $x$ and $h$, we need to try to express $S$ with only one variable. From the information given, we have the volume of the box = 10m$^3$, which means:

$x \times 2x \times h = 10$ or $h = \frac{10}{2x^2}$.

Putting this value of $h$ in $C$ we get,

$C(x) = 20x^2 + \frac{36x \times 10}{2x^2}$. When we simplify this, $C(x) = 20x^2 + 180x^2$, which we need to minimize.
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**STEP 5:** Calculating the first derivative of \( C(x) \) we get,
\[
C'(x) = 40x - 180x^2
\]
To find the critical points, we need to equate \( S'(x) \) to zero.
\[
=> 40x - 180x^2 = 0
\]
\[
=> x^3 = \frac{180}{40}	ext{ or } x^3 = \frac{9}{2}
\]
This will give us \( x = \left(\frac{9}{2}\right)^{1/3} \). Solving this using a calculator we get \( x \approx 1.65 \).

**STEP 6:** To test whether this value gives us a minimum, we take out the second derivative:
\[
C''(x) = 40 + \frac{360}{x^3}
\]
If we put \( x = 1.65 \) in this, we get \( C''(x) = 40 + \frac{360}{1.65^3} \). Solving this using a calculator, we get \( C''(x) = 120.14 \), which is positive. Thus, we know that we have minimized \( C \).

**STEP 7:** But, we still haven’t completed the problem. We still need to find the cost of the material for the cheapest container.

After putting in \( x = 1.65 \) in our original cost function we get:
\[
C(1.65) = 20(1.65)^2 + 180(1.65)^1
\]
Using a calculator, we obtain the cost as \( C \approx 163.54 \).

Therefore, the cost of the material for the cheapest container given the dimensions is \( $163.54 \).

References: The following works were referred to during the creation of this handout: Khan’s Academy’s “Optimization: Sum of Squares”, Khan Academy’s “Optimization: Cost of Materials”, and Stewart Calculus: Early Transcendentals, 5th Edition.